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The $S = 1$ Relativistic Oscillator

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Following upon my previous paper about the $S = 0$ relativistic oscillator, I build up an oscillator-like system which one can call the $S = 1$ Proca oscillator. The Proca field function is obtained in the framework of the Bargmann–Wigner prescription and the interaction is introduced similarly to the $S = 1/2$ Dirac oscillator case regarded by Moshinsky and Szczepaniak. I obtain the intriguing rule of quantization $\mathcal{E} = \hbar\omega/2$ for the parity states $(-1)^j$ and $\mathcal{E} = \pm\hbar\omega(j + 1/2)$ for the parity states $-(-1)^j$. There are no radial excitations. I apply the above-mentioned procedure to the case of the two-body relativistic oscillator, too.

1. THE PROCA OSCILLATOR—PUZZLED QUANTIZATION

Here I continue the study of oscillator-like systems first undertaken by Moshinsky and Szczepaniak [1] for the $S = 1/2$ case. Extensions of this model to the case of a two-body problem and to other spins have been presented in refs. 2 and 3–5, respectively. A detailed consideration of the $S = 1/2$ case presented in ref. 6 demonstrated that this form of interaction is free of the problem known as the Klein paradox. From the formal point of view the oscillator-like interaction can be interpreted as the interaction with a linear electric field $E^i = \kappa r^i$ [7] through the term $\sigma_{\mu\nu}F^{\mu\nu}/2$.

My previous work [8–11] explored several interesting features of oscillator-like systems. For instance, in refs. 8 and 10 the possibility of an oscillator-like construct for arbitrary spin in the Dowker formalism [12] is proved. Some ideas providing a basis for the matrix construct of the Klein–Gordon oscillator [3] are presented in refs. 8 and 9. In ref. 11 the Bargmann–Wigner (BW) set of equations is considered, with an *antisymmetric* second-rank

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spinor being chosen as a field function. Such a description leads to the Kemmer–Dirac formalism [13] for the $S = 0$ particle. The interaction introduced in each of the BW equations in the form proposed by Moshinsky and Szczepaniak results in an oscillator-like equation with double degeneracy (in N, the principal quantum number) of the spectrum in the low-frequency limit cf. [3].

The aim of this paper is to consider the $S = 1$ Proca oscillator[3] with the interaction introduced in the same manner as in ref. 11, namely, I start from the Bargmann–Wigner equations with the non-gauge interaction obtained after the substitution $\partial_i \to \partial_i + ik\gamma^0 r^i$, $i = 1, 2, 3$,

$$\begin{cases} [i\gamma^\mu \partial_\mu - k\gamma^i\gamma^0 r^i - m]\Psi(x) = 0 \\ \Psi(x)[i(\gamma^\mu)^T \overleftarrow{\partial}_\mu - k(\gamma^i\gamma^0)^T r^i - m] = 0 \end{cases} \tag{1}$$

The $S = 1$ BW field function presents itself a *symmetric* spinor of the second rank (4×4 symmetric matrix); the derivative acts to the left in the second equation. So the field function obeys the Dirac oscillator equation in each of the indices.

The symmetric wave function is expanded in the complete set of γ-matrices[4]

$$\Psi_{\{\alpha\beta\}} = \gamma^\mu_{\alpha\delta} C_{\delta\beta} A_\mu + \sigma^{\mu\nu}_{\alpha\delta} C_{\delta\beta} F_{\mu\nu} \tag{2}$$

The obtained equations for A_μ and $F_{\mu\nu}$ are

$$\partial_\nu F^{\nu 0} = -\frac{m}{2} A^0 + \frac{k}{2}(r^i A^i) \tag{3a}$$

$$\partial_\nu F^{\nu i} = -\frac{m}{2} A^i + \frac{k}{2} r^i A^0 \tag{3b}$$

$$2mF_{i0} = (\partial_i A_0 - \partial_0 A_i) + 2k(r^j F_i{}^j) \tag{3c}$$

$$2mF_{jk} = (\partial_j A_k - \partial_k A_j) - 2k(r^j F^0{}_k - r^k F^0{}_j) \tag{3d}$$

Let me introduce $E^i = F^{i0}$ and $B^i = -1/2\epsilon^{ijk}F^{jk}$. Then, expressing the dependence of the wave function on t as $\exp(-i\mathscr{E}t)$, one can obtain the equations ($c = \hbar = 1$)

[3] I take the liberty of naming the equations obtained below the Proca oscillator since in the free case equations (3a)–(3d) are the well-known Proca equations [14].

[4] Taking into account the symmetry properties of the field function, it is sufficient to use only $\gamma_\mu C$ and $\sigma_{\mu\nu} C$ [15] in the considered case; C is used as the matrix of a charge conjugation. Compare with formula (4), the $S = 0$ case, in ref. 11.

$$\left\{\begin{array}{c}(\mathscr{E} + m)E^i + \sqrt{2}\epsilon^{ijk}p_j^+B^k = \dfrac{i}{2}(\mathscr{E} + m)A^i - \dfrac{i}{\sqrt{2}}p_i^+A^0\\ (\mathscr{E} - m)E^i + \sqrt{2}\epsilon^{ijk}p_j^-B^k = -\dfrac{i}{2}(\mathscr{E} - m)A^i + \dfrac{i}{\sqrt{2}}p_i^-A^0\\ 2(p_iE^i) = imA^0 - ik(r^iA^i)\\ 2m\epsilon^{ijk}B^k - 2k(r^iE^j - r^jE^i) = i(p_iA^j - p_jA^i)\end{array}\right. \tag{4}$$

where $\vec{p}^{\,\pm} = (1/\sqrt{2})(\vec{p} \pm k\vec{r})$ and $(\vec{p}_i) = (1/i)\vec{\nabla}_i$. This set can be rewritten in a more symmetric form after the substitution $D^i = E^i - (i/2)A^i$ and $F^i = E^i + (i/2)A^i = (D^i)^*$ if E^i and A^i are real quantities. In such a way one obtains

$$\left.\begin{array}{c}\dfrac{im}{\sqrt{2}}A^0 = p_i^+F^i + p_i^-D^i\\ mB^i = \dfrac{1}{\sqrt{2}}\epsilon^{ijk}[p_j^+F^k - p_j^-D^k]\\ (\mathscr{E} + m)D^i = -\sqrt{2}\epsilon^{ijk}p_j^+B^k - \dfrac{i}{\sqrt{2}}p_i^+A^0\\ (\mathscr{E} - m)F^i = -\sqrt{2}\epsilon^{ijk}p_j^-B^k + \dfrac{i}{\sqrt{2}}p_i^-A^0\end{array}\right\} \tag{5}$$

It is possible to eliminate A^0 and B^i on using the commutation relations $[p_i^+, p_j^-]_- = ik\delta_{ij}$ and $\{p_i^+p_j^- - p_j^+p_i^-\}f(\vec{r}) = k\epsilon^{ijk}\hat{L}^kf(\vec{r}) = ik(\vec{S}\vec{L})_{ij}f(\vec{r})$. The result is

$$m(\mathscr{E} + m)\, D^i = [-ik(\vec{S}\vec{L})_{ij} - \vec{p}_k^{\,+}\vec{p}_k^{\,-}\delta_{ij}]D^j + [2(\vec{S}\vec{p}^{\,+})^2_{ij} - (\vec{p}^{\,+})^2\delta_{ij}]F^j \tag{6a}$$

$$m(\mathscr{E} - m)F^i = [(\vec{p}^{\,-})^2\delta_{ij} - 2(\vec{S}\vec{p}^{\,-})^2_{ij}]D^j + [-ik(\vec{S}\vec{L})_{ij} + \vec{p}_k^{\,-}\vec{p}_k^{\,+}\delta_{ij}]F^j \tag{6b}$$

$\vec{S}$ are the spin-1 matrices, $\vec{L}$ is the orbital part of the angular momentum operator.

In order to carry out further decoupling one could try to apply the procedure of ref. 11 to the system of equations (6a)–(6b). But the calculations are more complicated compared with the previous work and do not lead directly to the desired result. The system is *not* decoupled after the first application of the procedure of ref. 11. We arrive at

$$m^2(\mathscr{E}^2 - m^2)D^i = [\text{dif.op.}_1]_{ij}D^j - 2ikp_i^+p_j^+F^j \tag{7a}$$

$$m^2(\mathscr{E}^2 - m^2)F^i = [\text{dif.op.}_2]_{ij}F^j + 2ikp_i^-p_j^-D^j \tag{7b}$$

with complicated operators $\text{dif.op}_{1,2}$ on the right-hand side of the equation. On the other hand, we do not want to apply the procedure of ref. 18, p. 298, because it is doubtful that we can insert the complete set of the state vectors as in the formulas (108), (109) of the cited reference between $\boldsymbol{\eta}\cdot\mathbf{S}$ and $\boldsymbol{\xi}\cdot\mathbf{S}$.

Such bra vectors as in (108a), (108b) may *not* exist, e.g., in the case of low quantum numbers N and j.

Nevertheless, one can use another method. Namely, (1) multiplying, e.g., the first equation (6a) by $(D^i)^*$, (2) integrating over $d^3\mathbf{r}$, and (3) using identities of the Hermitian conjugation, expansion in spherical tensors, and normalization conditions, the problem is solved. On this basis we derive the quantization rule for purely imaginary $k = im\omega$:

$$\mathscr{E} = -\frac{ik}{2m}\,[j(j+1) - l(l+1) - s(s+1)] - \frac{3ik}{2m} \tag{8}$$

Very surprisingly, the principal quantum number is *not* present here! The energy is due to the spin–orbit interaction and the constant term, which is similar to that which appeared in the Moshinsky–Szczepaniak version of the $S = 1/2$ oscillator. Finally, for the states of parity $(-1)^j$ one has

$$\mathscr{E} = \hbar\omega/2 \tag{9}$$

which can be interpreted as zero-mode oscillations (we recover $\hbar$ for visual purposes). On the other hand, for the states of parity $-(-1)^j$ one has

$$\mathscr{E} = \pm\frac{\hbar\omega}{2}\,(2j+1) \tag{10}$$

i.e., the *nonrelativistic* formula for the harmonic oscillator with the substitution $N \to j$ and with two signs of the energy![5] We do not know any appropriate Hamiltonian for higher spins. That proposed by Weaver *et al.* [16] leads to *nonlocal* theory even in the interaction-free case. Therefore, one can only speculate about the origin of the established infinite degeneracy in the N quantum number. We must leave the detailed interpretation of Eqs. (9), (10) for future publications.

2. THE TWO-BODY RELATIVISTIC OSCILLATOR

Now I consider the case of the two-body Dirac oscillator [2]. The two-body Dirac Hamiltonian with oscillator-like interaction is given by ($m\omega = 1$ and, hence, $k = i$ here)[6]

$$i\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)\psi = \mathscr{H}\psi$$

$$= \Bigg[\frac{1}{\sqrt{2}}\,(\vec{\alpha}_1 + \vec{\alpha}_2)\cdot\vec{P} + \frac{1}{\sqrt{2}}\,(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p}$$

[5] Compare with the discussion on p. 177 in ref. 11.

[6] The oscillator-like system with $\sim(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{r}B\Gamma_5$ has also been considered.

$$- \frac{i}{\sqrt{2}} (\vec{\alpha}_1 - \vec{\alpha}_2) \cdot \vec{r} B + m(\beta_1 + \beta_2) \Big] \psi \tag{11}$$

where the matrices are given by the direct products

$$\begin{aligned} \vec{\alpha}_1 &= \begin{pmatrix} 0 & \vec{\sigma}_1 \\ \vec{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{2\otimes 2} & 0 \\ 0 & \mathbb{1}_{2\otimes 2} \end{pmatrix}, \\ \vec{\alpha}_2 &= \begin{pmatrix} \mathbb{1}_{2\otimes 2} & 0 \\ 0 & \mathbb{1}_{2\otimes 2} \end{pmatrix} \otimes \begin{pmatrix} 0 & \vec{\sigma}_2 \\ \vec{\sigma}_2 & 0 \end{pmatrix} \end{aligned} \tag{12}$$

$$B = \beta_1 \otimes \beta_2 = \begin{pmatrix} \mathbb{1}_{2\otimes 2} & 0 \\ 0 & -\mathbb{1}_{2\otimes 2} \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{2\otimes 2} & 0 \\ 0 & -\mathbb{1}_{2\otimes 2} \end{pmatrix} \tag{13}$$

$$\Gamma_5 = \gamma_1^5 \otimes \gamma_2^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\otimes 2} \\ \mathbb{1}_{2\otimes 2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathbb{1}_{2\otimes 2} \\ \mathbb{1}_{2\otimes 2} & 0 \end{pmatrix} \tag{14}$$

If we are in the center-of-mass system (c.m.s.) it is possible to equate $\vec{P} = 0$. While the two-body Dirac oscillator seems not to have found considerable phenomenological applications (see spectra presented in ref. 18 in connection with experiment), this is an interesting mathematical model.

Now I apply the same procedure as that used for the transformation of the Bargmann–Wigner set of equations to the Proca equations (see refs. 11 and 15 and above): the 16-component wave function of two-body Dirac equation can also be expanded in the complete set of matrices: $(\gamma^\mu C)$, $(\sigma^{\mu\nu} C)$ and C, $(\gamma^5 C)$, and $(\gamma^5 \gamma^\mu C)$. The wave function is decomposed into symmetric and antisymmetric parts using the above-mentioned complete system of matrices

$$\psi = \psi_{\{\alpha\beta\}} + \psi_{[\alpha\beta]} \tag{15}$$

where the first term is given by formula (2) and the second term by formula (4) of ref. 11. In such a way we obtain the set of equations[7]

$$\mathscr{E} A_0 = 0, \qquad \mathscr{E} \tilde{A}_0 = -2m\tilde{\varphi} \tag{16a}$$

$$\mathscr{E} \varphi = 4i \vec{p}_i^{\,-} F^{i0} \tag{16b}$$

$$\mathscr{E} \tilde{\varphi} = -2m\tilde{A}_0 + 2\epsilon^{ijk} \vec{p}_i^{\,+} F^{jk} \tag{16c}$$

$$\mathscr{E} \tilde{A}^i = 2i\epsilon^{ijk} \vec{p}_j^{\,\mp} A^k \tag{16d}$$

$$\mathscr{E} A^i = 4imF^{0i} + 2i\epsilon^{ijk} \vec{p}_j^{\,\pm} \tilde{A}^k \tag{16e}$$

[7] I correct here the misprints in the signs of the equations of ref. 8.

$$\mathcal{E}F^{0i} = -2imA^i + 2i\vec{p}_i^{\,+}\varphi \tag{16f}$$

$$\mathcal{E}F^{jk} = \epsilon^{ijk}\vec{p}_i^{\,-}\tilde{\varphi} \tag{16g}$$

The signs in the set (16a)–(16g) correspond to two types of Dirac oscillator-like interactions, with $\sim(\vec{\alpha}_1 - \vec{\alpha}_2)B$ and $\sim(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5$, respectively.

The two-body Dirac oscillator equations in the form (16a)–(16g) can be decoupled into a set containing only functions φ, $\tilde{\varphi}$, and $\tilde{A}_\mu$ and another one containing only A_μ and $F_{\mu\nu}$:

$$(\mathcal{E}^2 - 8m^2)\varphi = 8(\vec{p}_i^{\,-}\vec{p}_i^{\,+})\varphi - \frac{16im}{\mathcal{E}}\epsilon^{ijk}\vec{p}_i^{\,-}\vec{p}_j^{\,\pm}\tilde{A}^k \tag{17a}$$

$$(\mathcal{E}^2 - 4m^2)\tilde{\varphi} = 4(\vec{p}_i^{\,+}\vec{p}_i^{\,-})\tilde{\varphi} \tag{17b}$$

$$\mathcal{E}\tilde{A}_0 = -2m\tilde{\varphi} \tag{17c}$$

$$(\mathcal{E}^2 - 8m^2)\tilde{A}^i = 4(\vec{p}_j^{\,\mp}\vec{p}_j^{\,\pm})\tilde{A}^i - 4(\vec{p}_j^{\,\mp}p_i^{\pm})\tilde{A}^j - \frac{16im}{\mathcal{E}}\epsilon^{ijk}\vec{p}_j^{\,\mp}\vec{p}_k^{\,+}\varphi \tag{17d}$$

and

$$\mathcal{E}A_0 = 0 \tag{18a}$$

$$\mathcal{E}^2A^i = 4im\mathcal{E}F^{0i} + 4(\vec{p}_j^{\,\pm}\vec{p}_j^{\,\mp})A^i - 4(\vec{p}_j^{\,\pm}\vec{p}_i^{\,\mp})A^j \tag{18b}$$

$$\mathcal{E}^2F^{0i} = -2im\,\mathcal{E}\,A^i - 8(\vec{p}_i^{\,+}\vec{p}_j^{\,-})F^{0j} \tag{18c}$$

$$(\mathcal{E}^2 - 4m^2)F^{jk} = 2\epsilon^{ijk}\epsilon^{lmn}(\vec{p}_i^{\,-}\vec{p}_l^{\,+})F^{mn} \tag{18d}$$

This proves that the Dirac oscillator interaction, like the case when we introduce the (self-) interaction with the transverse 4-vector potential into the Proca equation (or, equivalently, into the Bargmann–Wigner equations), does not mix $S = 1$ and $S = 0$ states.

The solutions of equations for the two-body relativistic oscillator have been given in ref. 17, see the formulas (62.20), (62.27), (62.33) and (62.23), (62.30), (62.35), (62.43), (62.45), (62.49). However, the comparison is rather difficult because instead of the second-rank spinors for $S = 0$ and $S = 1$ I use the corresponding scalar, 4-vector, and antisymmetric tensor field functions (Sankaranarayanan and Good [16]).

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